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Decoding the Most Common Errors in Combined Operations with Integers

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Communication Summary

Combined operations arise when multiple mathematical operations must be performed within a single expression. It is essential to reflect on the conventions that determine the order of operations and the strategies used to record the process.

During the 2023–2024 academic year, the Vèrtex research team, led by Innovamat, analyzed the work of one thousand students from 22 schools who solved 12 combined operations. The goal was to identify recording strategies and detect the most common errors among 7th grade students. This communication presents, through concrete examples, the main strategies observed and the typical associated errors, with the aim of providing educators with tools to identify and address difficulties related to understanding the hierarchy of operations and problem-solving.

KEY WORDS: combined operations – recording strategies – errors

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1. Introduction

Combined operations occur whenever two or more sequential operations must be performed and expressed within a single expression. This type of calculation requires a deep understanding of the hierarchy of operations and often presents challenges for middle school students. It is particularly important to reflect on both the convention that determines the order of operations and the strategies used to record the process.

The currently accepted hierarchy, widely used today, was not formalized until less than a century ago and is the result of an arbitrary agreement. To avoid placing parentheses around every operation, a convention was established—though it could have been different. Regardless, understanding this hierarchy is essential, both from a mathematical content perspective and from a process-oriented viewpoint, as it involves recognizing the need for conventions that, while not logically deducible, are crucial for effective mathematical communication.

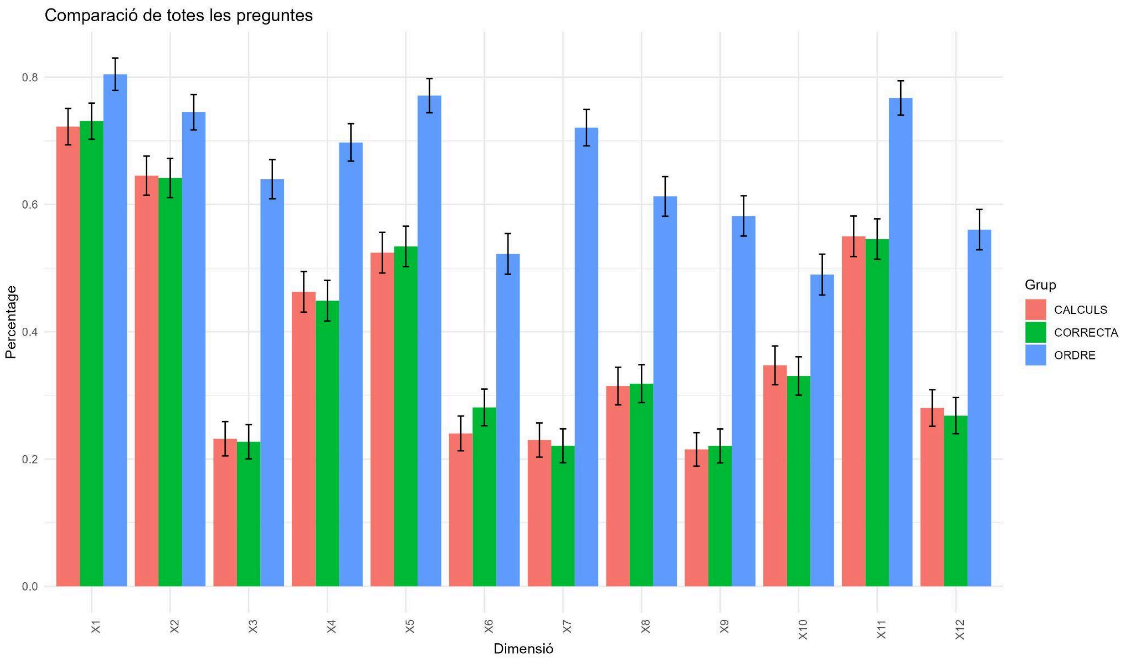
Once this hierarchy is established, a wide range of recording strategies becomes possible. To analyze these strategies and understand the errors that may arise from them, the Vèrtex research team, led by Innovamat, conducted the study that forms the basis of this communication.

$49 - 32 : 8 - 11$	$11 + 35 : (7^2 - 44)$	$(-3)^2 + 4 \cdot (31 - 27)$
$36 + 4 \cdot (-2) + 12$	$-15 + (3^2 + 5) : 7$	$25 \cdot (-2)^2 : (-4 - (-6))$
$6 - 60 : (-8 + 2)$	$-6 - 7 \cdot (-4) + 10$	$(-2)^3 + 11 \cdot (-7 + (-1))$
$18 + 10 - (18 - 16)^2$	$16 + (-4 + 14) \cdot 3$	$-9 \cdot (-4) - (-6) + 10$

Sample of the 12 combined operations analyzed.

2. Errors in Solving Combined Operations

An initial analysis of the data revealed that a high percentage of participants followed the correct operational order when solving the problems. However, the percentage of students who performed the calculations correctly was significantly lower.



Graph showing the percentage of participants (y-axis) who answered correctly for each test item (x-axis), in each of the analyzed categories: (1) in red, if the calculations are correct; (2) in blue, if the order followed is correct; and (3) in green, if the final solution is correct.

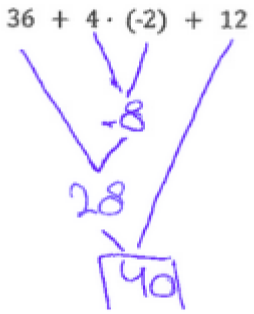
Thus, the primary cause of errors in solving combined operations in this study was related to calculation mistakes, such as confusion with integer operations or difficulties in calculating powers. Nevertheless, a considerable number of participants also made errors related to the order of operations or the way they recorded their solutions. These types of errors are examined in greater detail in this communication.

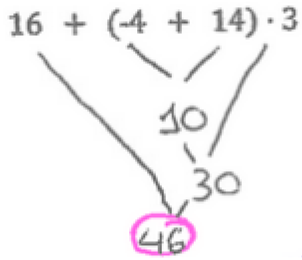
3. **Recording Strategies Observed**

The collected responses reveal a wide variety of strategies used to indicate the process followed in solving a combined operation. These strategies can be grouped into four categories:

3.1. **Tree Strategy**

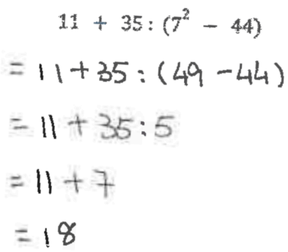
This strategy involves breaking down the operation into smaller hierarchical steps using lines that connect the numbers involved in each partial calculation. At the convergence points of these lines, the partial results are indicated, leading ultimately to the final result.

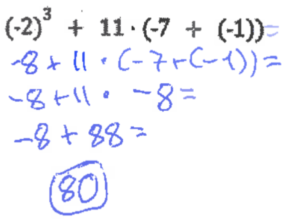




3.2. **Sequential Strategy**

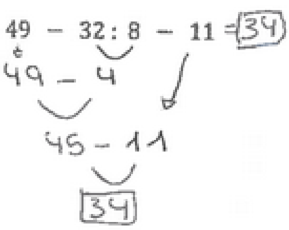
The sequential strategy involves rewriting the combined operations across multiple lines, one below the other, progressively reducing the number of operations until obtaining the final result[1]:

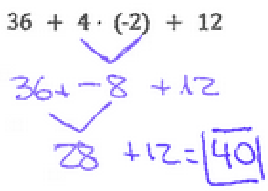




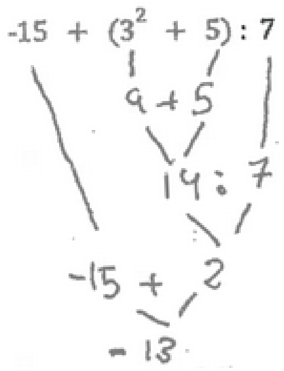
3.3. **Mixed Strategy**

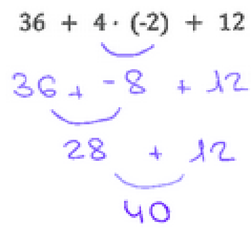
In this approach, students combine elements of both the tree and sequential strategies. While lines are used to indicate which part of the operation is solved first, the operation is also rewritten.





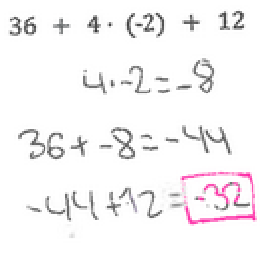
[1] In the second example presented below, as well as in others that follow, there are calculation errors. However, these did not lead us to discard the student's work, as it provides meaningful insight into the strategy used to record the order in which the operations were performed.

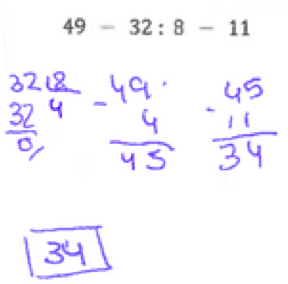


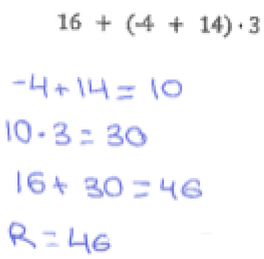


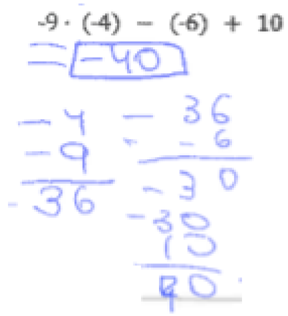
3.4. **Fragmented Strategy**

Students employing this strategy present a more fragmented and less structured solution. Their recording focuses on isolated, simple operations solved at each step, without a clear or complete representation of the overall expression.

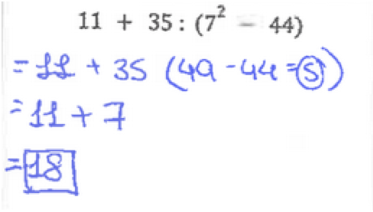


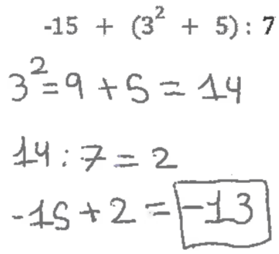






In reviewing the use of sequential and fragmented recording strategies, we observed that some students made incorrect use of the equal sign (=).

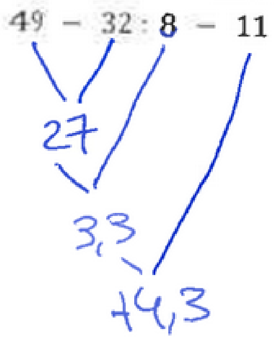


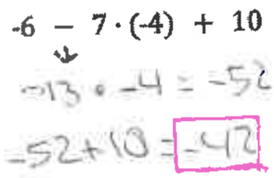


4. **Examples of Types of Errors Observed**

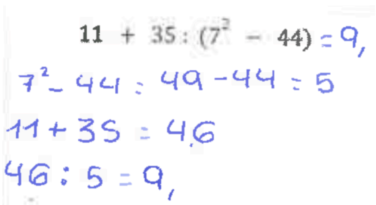
As previously mentioned, a significant portion of errors in final results stem from calculation mistakes—particularly mental calculations, given the magnitude of the numbers involved. However, some students still exhibit difficulties with the order of operations. Below are the main types of errors identified:

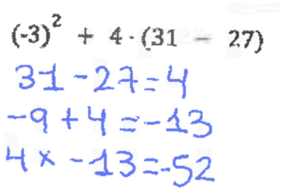
4.1. Errors derived from operating from left to right



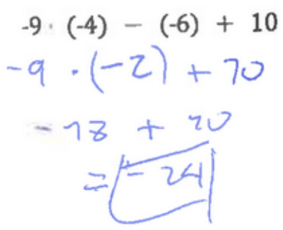


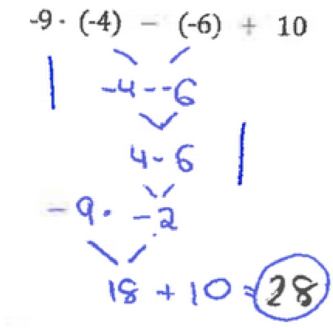
4.2. Errors derived from prioritizing parentheses but ignoring hierarchy among remaining operations.



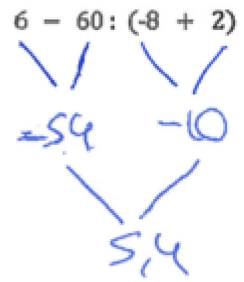


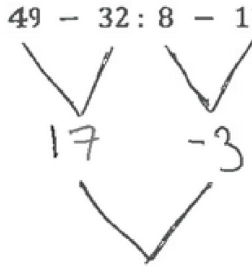
Regarding the use of parentheses to indicate priority, we observed that some students may overlook the ambiguity of this symbol. Parentheses serve a dual purpose: they help avoid confusion with negative signs and indicate which operations should be performed first. This dual function can complicate interpretation, as illustrated in the following examples:



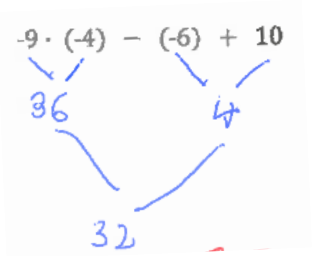


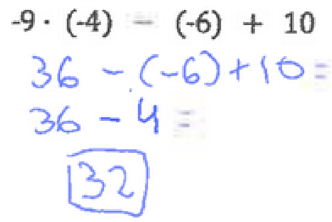
4.3. Errors derived from prioritizing additive operations over multiplicative ones.

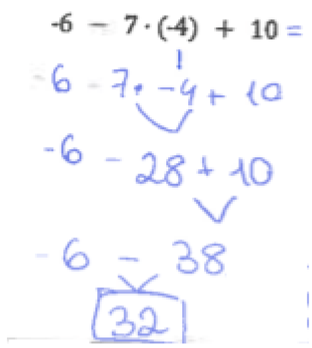


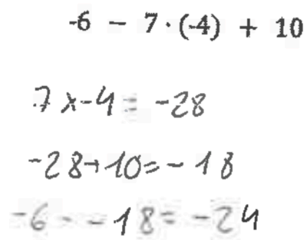


4.4. Errors derived from not operating from left to right when there are operations of the same hierarchy.









At times, not strictly following the rule of performing operations from left to right—when the operations are of the same hierarchical level—can still lead to a correct result. The challenge in these cases is that, based solely on the written work, it's difficult to determine whether the student made a deliberate choice—recognizing that the order wouldn't affect the outcome and selecting a strategy that simplifies the calculation—or whether the choice was arbitrary and lacked mathematical reasoning (Kontorovich & Zazkis, 2017).

It is important to note that such flexibility, when based on informed reasoning, is highly desirable. As Bay-Williams and SanGiovanni (2021) emphasize, fostering strategic flexibility in calculations is key to developing a deeper understanding of operations and the mathematical system as a whole.

5. Conclusions and Educational Implications

The study's findings reveal that a significant portion of the errors made by students are not random but instead reflect patterns of thinking and strategies developed in the absence of a clear model for solving combined operations. Understanding these strategies is fundamental for designing more targeted and effective pedagogical interventions.

The analysis of specific cases highlights the importance of moving beyond repetitive practice when teaching combined operations. It is crucial to incorporate opportunities for metacognitive reflection on the steps taken, the ways in which they are recorded, and the decisions made throughout the problem-solving process.

The study underscores the need to make the hierarchy of operations explicit and visible in mathematics instruction. It also demonstrates how identifying and analyzing specific errors can help educators better align their teaching practices with the actual needs of their students.

In this regard, understanding students' strategies and associated errors enables educators to:

- Design activities that promote awareness of the procedures used.
- Introduce meaningful classroom discussions about the meaning of the equal sign and the various ways mathematical work can be recorded.
- Reflect together on when it is possible—and even convenient—to make the established order of operations more flexible.

6. Bibliography

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Kontorovich, I., & Zazkis, R. (2017). Mathematical conventions: Revisiting arbitrary and necessary. *For the Learning of Mathematics*, 37(1), 29-34